Dynamic Stability of a Translating Vehicle with a Simple Sling Load

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A system consisting of helicopter with a singly tethered load is modeled as two particles connected by an inextensible link. The nonlinear equations of motion are linearized about a steady level flight condition, and it is shown that the perturbation equations separate into longitudinal and lateral/directional sets. For some selected parameter values, the modes are extracted and interpreted. Generally, instability will arise if the mass of the load is greater than the mass of the helicopter, or, if the tether length exceeds a critical value. Typically, this critical length is far in excess of reasonable values.

Nomenclature

b = ratio of ballistic coefficients of helicopter and load
$D_{L\xi}, D_{L\eta}, D_{L\zeta}$ = components of aerodynamic drag on load
$\hat{e}_t, \hat{e}_n, \hat{e}_o$ = components of thrust force on helicopter
g = acceleration due to gravity
K_H, K_L = drag constants of helicopter and load, respectively
ℓ = length of tether from helicopter to load
M, m = masses of helicopter and load, respectively
R = magnitude of restraint force in tether
s = generic variable in characteristic polynomial
v, v_L = speeds of helicopter and load, respectively
x, y, z = Cartesian coordinates of helicopter
$\boldsymbol{x}$ = general state vector
γ = flight-path angle of helicopter
ξ, η, ζ = Cartesian coordinates of load
θ, λ = pendular polar and azimuthal angles, respectively
ϕ = heading angle of helicopter
u, δ, ϵ = perturbations in V, θ, and λ, respectively
q, r = $\dot{\delta}$ and $\dot{\lambda}$, respectively

Superscripts

(e) = equilibrium values
$\hat{()}$ = dimensionless parameter
$\dot{()}$ = time derivative

Introduction

PERHAPS the most attractive feature of the heavy-lift helicopter is its ability to carry large and bulky objects as externally slung loads. This is particularly important in transportiong loads over rough terrain, where land vehicles cannot go and conventional aircraft cannot land. Such helicopters can also be used as hoists for construction in remote or inaccessible areas. However, if the full potential of the heavy-lift helicopter is to be realized, then it must be sure that the helicopter-load configuration is stable and safe. In this paper we analyze the motion of a simplified model of a helicopter in uniform motion with slung load.

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The helicopter/load system may be characterized as two bodies connected by tethers with each body subject to aerodynamic forces. In the framework of analytical mechanics, a mathematical model may run the gamut from reasonably simple to extremely complex, according to the level of detail in the description of the bodies, tethers and aerodynamics. For example, the helicopter or the load may be represented as a point mass, a rigid body, a rigid body with spinning components, or a flexible body. The tethers may be single or multiple and inextensible or elastic. Finally, the aerodynamics might be completely neglected, or treated as complete six-component models for each body, including aerodynamic forces on the tethers and downwash from the helicopter rotor. If in each case, one chooses the most complex component description, then the combined model will probably be very accurate but it will certainly involve a bewildering array of parameters. With such a complicated model, a systematic search for instabilities would be very costly, and the task of understanding and curing them would be very difficult. For these purposes, simplification is essential.

In view of these comments about model complexity, it may be suggested that great simplification is realized if one neglects helicopter motions and considers the load only. Thus, a considerable amount of work has been done on aerodynamic instabilities of a tethered object in a windstream.[1-5] While this is an important first step in predicting helicopter/load instabilities, it should not be inferred that freedom from such aerodynamic instability is sufficient to ensure a stable helicopter/load configuration. When the mass or aerodynamic force of the load are comparable to those of the helicopter, then one should not ignore motions of the helicopter induced by the load. Thus, models including vehicle/load interaction must be studied.

Studies have been made[6-9] of relatively detailed models, including helicopter/load interactions. References 6 and 7 report results of flight tests, and so are more suited to evaluation than prediction. In Ref. 8, Abzug derived linearized equations for a model with two rigid bodies joined by two rigid tethers. The system is linearized about a steady, level flight condition (possibly hover) but aerodynamics are not included. Some analysis of the modes of motion is discussed but no instabilities are predicted. Liu[9] extended Abzug's linearized equations to include aerodynamics. Stability analyses and simulations were performed for a CH-47 vehicle carrying several typical loads. Although stability charts are presented showing the effects of some system parameters, other important parameters are ignored. For instance, the ratio of the mass of the load to that of the vehicle is not discussed. The present analysis reveals that this mass ratio is a very important parameter. Finally, Lucassen and Sterk[10] consider the stability of vertical plane motions of a helicopter/load system in hover. They are able to identify a helicopter mode and a load mode, however, the majority of the results are for particular parameter values.

The principal new feature of the present analysis is the inclusion of three-degree-of-freedom helicopter motions which are induced by oscillations of the load. In the next section the model is described and the method of deriving the equations is sketched. A more detailed explanation is covered in the Appendix. Following this, the equations are linearized about a steady, level flight condition and it is observed that the perturbation equations separate into longitudinal and lateral sets. The stability of the motions is analyzed and some results are presented. Finally, the practicability of the predicted instabilities is discussed.

Equations of Motion

The model of the helicopter-load system is shown in Fig. 1. Both helicopter and load are considered to be point masses connected by a rigid link of negligible mass. The forces acting on each body include its weight, an aerodymamic force and the restraint from the link. The aerodynamic force is drag only (i.e., no lift or side force) and is given by the usual square-speed law. No downwash on the load is included. Additionally, the helicopter has a thrust force, as shown.

To derive the equations, Newton's second law is applied to each body separately and then the unknown restraint force is eliminated. For the helicopter it is convenient to use flight path coordinates‡ with unit vectors: $\hat{e}_t$ tangential to the flight path; $\hat{e}_n$ normal to the flight path in the vertical plane; and $\hat{e}_o$ normal to the flight path in the horizontal plane. The equations of motion for the load are written in terms of components along Cartesian axis with coordinates (ξ, η, ζ). These load coordinates are related kinematically to the Cartesian coordinates of the helicopter (x, y, z) and the pendulum polar and azimuthal angles θ and λ, respectively. The unknown restraint force R is eliminated from the set of equations and the kinematic relations are used to express the components of the load velocity and acceleration in terms of the variables $(v, \gamma, \theta, \lambda, \phi)$. Details are given in the Appendix.

Stability Analysis

The analysis of stability requires that the equations be linearized about some equilibrium point. The equilibrium point of interest has the longitudinal variables $v = v^e$, $\gamma = 0$, $q = 0$, $\theta = \theta^e$, while the lateral variables have $\phi = 0$, $\lambda = \pi$. If these values are substituted into the dynamical equations (A10-A16) it is found that for equilibrium it is necessary that:

$$F_t^e = (K_H + K_L) v^{e2} \tag{1}$$

$$F_n^e = (M+m) g \tag{2}$$

‡Thus, the analysis is not suitable for the hover condition.

and

$$\tan\theta^e = K_L v^{e2} / mg \tag{3}$$

The obvious physical interpretations of these equations require no comment.

To examine the stability of the equilibrium, the dynamical equations will now be linearized. Before doing so it is convenient to introduce dimensionless variables. We take v^e as unit speed, g as unit acceleration and M as unit mass. It follows then that v^{e2}/g is the unit distance; v^e/g is the unit time, and; Mg is the unit force.

Since q, γ, and ϕ are zero at equilibrium there is no confusion if the same symbols are used for the perturbed quantities.§ The perburbations in v and θ will be denoted by u and δ, respectively, while the perturbations in λ is denoted by ϵ. *No perturbation in thrust (F_t, F_n, F_o) is considered.* The system equations are expanded and the equilibrium condition used¶ to obtain:

$$(M+m)\dot{u} + 2v^e(K_H + K_L)u + (M+m)g\gamma - m\ell\cos\theta^e\dot{q} - 2K_L v^e \ell\cos\theta^e q = 0 \tag{4}$$

$$(M+m)v^e\dot{\gamma} + m\ell\sin\theta^e\dot{q} + K_L v^e \ell\sin\theta^e q = 0 \tag{5}$$

$$(M+m)v^e\dot{\phi} - m\ell\sin\theta^e\dot{r} - K_L v^e\ell\sin\theta^e r = 0 \tag{6}$$

$$m[v^e\dot{\phi} - \ell\sin\theta^e\dot{r}] - K_L v^e[v^e(\epsilon - \phi) + \ell\sin\theta^e r] = 0 \tag{7}$$

$$m[-\cos\theta^e\dot{v} + v^e\sin\theta^e\gamma + \ell\dot{q}] + mg\cos\theta^e\delta + K_L v^e[v^e\sin\theta^e(\delta+\gamma) - 2\cos\theta^e u + \ell(1+\cos\theta^e)q] = 0 \tag{8}$$

Of course, we also have the kinematical equations

$$\dot{\epsilon} - r = 0 \tag{9}$$

$$\dot{\delta} - q = 0 \tag{10}$$

The most important observation concerning the perturbation equations (4-10) is that *the lateral and longitudinal motions are uncoupled.* Only Eqs. (6, 7, and 9) involve the lateral motion, while the longitudinal variables appear in Eqs. (4, 5, 8, and 10). The perturbation equations for the longitudinal motion can be put in the form $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ where $\boldsymbol{x} = [u, \gamma, q, \delta]^T$ and A is given below

$$A = \begin{bmatrix} -2\tan\theta^e[b + \hat{m} - \hat{m}(1-b)\cos^2\theta^e]/(1+\hat{m}) & -(1+\hat{m}) & -\hat{m}\ell^3\sin^3\theta^e/(1+\hat{m}) & -\hat{m} \\ -2(1-b)\hat{m}\sin^2\theta^e/(1+\hat{m}) & \hat{m}\tan\theta^e & \hat{m}\sin^2\theta^e\cdot\cos\theta^e/(1+\hat{m}) & \hat{m}\tan\theta^e \\ 2(1-b)\sin\theta^e/\hat{\ell} & -(1+\hat{m})/\hat{\ell}\cos\theta^e & -\tan\theta^e(1+\cos\theta^e) & -(1+\hat{m})/\hat{\ell}\cos\theta^e \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note that there are four parameters in A: $\hat{m}$ is the mass ratio; $\hat{\ell}$ is the dimensionless tether length; θ^e is the equilibrium polar angle; and b is the ratio of the ballistic coefficients of the helicopter and load $[b = (K_H/M) / (K_L/m)]$. From Eq. (9) it is seen that tan θ^e is equal to the drag to weight ratio

§All quantities are nondimensionalized as above.

¶With $F_o = 0$, the equilibrium conditions for the lateral motion are satisfied identically.

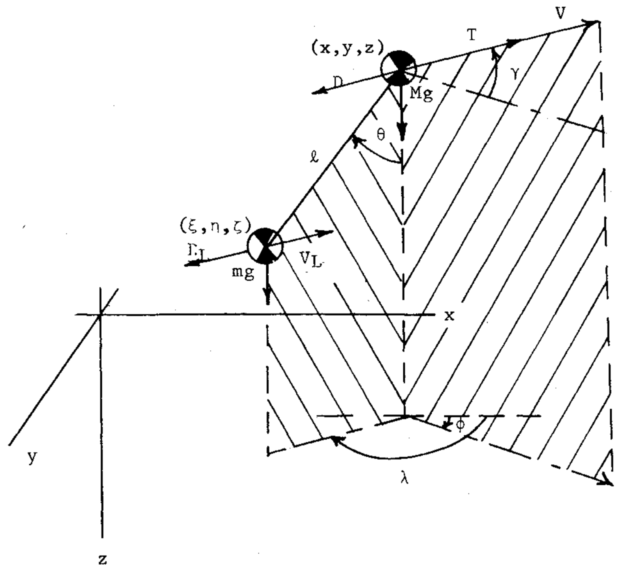

Fig. 1 Model of helicopter with slung load.

(D/W) for the load. Since the system matrix contains four parameters a general stability analysis is not practical. Indeed, the presentation of stability charts would be tedious and not particularly illuminating. We shall examine the results for certain special cases.

Case I ($b=1$): In this case, the load and helicopter have the same ballistic coefficient, and there results considerable simplification in the system matrix. The characteristic equation is

$$s(s+2\tan\theta^e)\{s^2+s\tan\theta^e(1+\cos^2\theta^e-\hat{m}) + (1+\hat{m})/\hat{\ell}\cos\theta^e-\hat{m}\tan^2\theta^e\}=0 \tag{11}$$

The zero eigenvalue corresponds to a rigid motion of the system with eigenvector $[1, -2\tan\theta^e, 0, -2\tan\theta^e]^T$. Thus, the helicopter can speed up ($\hat{u}>0$) and descend ($\gamma<0$) while the load maintains the same angle relative to the helicopter's velocity vector.

A second eigenvalue is $s=-2\tan\theta^e$ which is negative and therefore stable. The mode shape is $(1\ 0\ 0\ 0)^T$. This arises because the helicopter and load have precisely the same ballistic coefficient. Thus, a change in speed produces identical accelerations of helicopter and load.

The final pair of eigenvalues are found from the solution of the quadratic, and in general their stability depends on the parameter values: θ^e, $\hat{m}$, and $\hat{\ell}$. Necessary and sufficient conditions for stability are

$$\hat{m}<1+\cos^2\theta^e \tag{12}$$

and

$$\hat{\ell}<\hat{\ell}_c=(1+\hat{m})/\hat{m}\sin\theta^e\tan\theta^e \tag{13}$$

If condition (13) is not satisfied, a static instability arises; that is, if arbitrarily displaced from equilibrium, the system simply diverges. On the other hand, if Eq. (12) is not fulfilled, a dynamic instability can arise; that is oscillations grow in amplitude. Figure 2 shows a stability chart illustrating acceptable values of $\hat{m}$ and $\hat{\ell}$ for two values of D/W (i.e., $\tan\theta^e$). Figure 3 is a graph of the critical length ($\hat{\ell}_c$) as a function of (D/W) for $\hat{m}=0.5$.

Case II ($\theta^e=0$): θ^e equals zero is the limiting case of low drag-to-weight ratio for the load. The characteristic equation of the system becomes

$$s^2[s^2+(1+\hat{m})/\hat{\ell}]=0 \tag{14}$$

The repeated root s equals zero, has geometric multiplicity[11] one, and the eigenvector is $[1, 0, 0, 0]^T$. With no

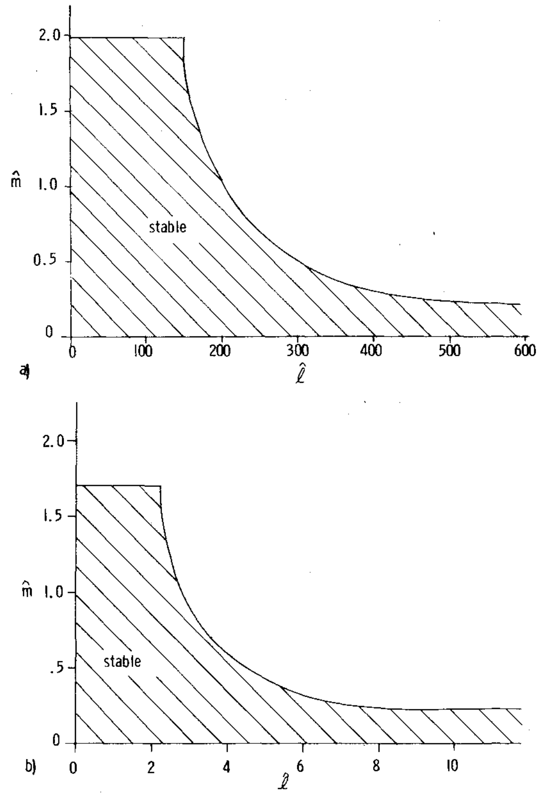

Fig. 2 Longitudinal stability chart; $b=1$: a) $(D/W)=0.1$; b) $(D/W)=1.0$.

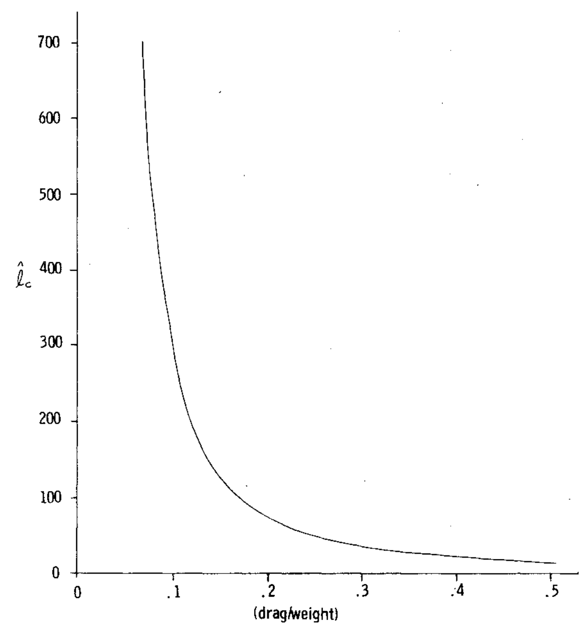

Fig. 3 Critical length as a function of drag to weight ratio; $\hat{m}=0.5$.

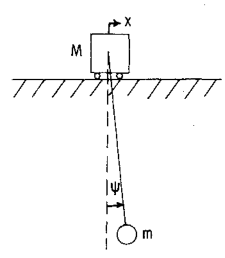

Fig. 4 Equivalent system; $(D/W)=0$.

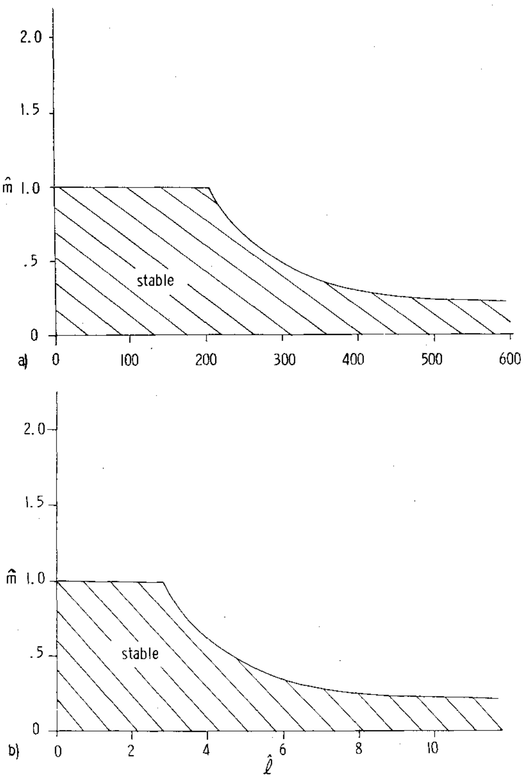

Fig. 5 Lateral stability chart: a) $(D/W)=0.1$; b) $(D/W)=1.0$.

aerodynamic force, speed disturbances do not affect the system.

The remaining roots are an imaginary pair with complex eigenvector $\{(-i\hat m/[(1+\hat m)/\hat\ell]^{1/2}, 0, i[(1+\hat m)/\hat\ell]^{1/2}, 1)\}^T$. There is no flight path oscillation, however, the angular velocity leads the angular displacement by 90°. The motion, at frequency $[(M+m)g/M\ell]^{1/2}$, is exactly the same as that of the system shown in Fig. 4.

In order to examine the lateral stability, we again seek the first-order normal form of the perturbation equations. Accordingly, as before, we put the equations in (dimensionless) normal form ($\dot x = Ax$). "x" is the three component vector $x=[\phi, r, \epsilon]^T$ and A is given below.

$$A=\begin{bmatrix} \hat m\tan\theta^e & 0 & -\hat m\tan\theta^e \\ (1+\hat m)/\hat\ell\cos\theta^2 & -\tan\theta^e & -(1+\hat m)/\hat\ell\cos\theta^e \\ 0 & 1 & 0 \end{bmatrix}$$

The stability of the lateral motion is determined by the eigenvalues of A, which has characteristic equation

$$s[s^2+\tan\theta^e(1-\hat m)s+(1+\hat m)/\hat\ell\cos\theta^e-\hat m\tan^2\theta^e]=0 \quad (15)$$

The root s equals zero has eigenvector [1 0 1] and corresponds to a rigid rotation of the helicopter and load. Clearly, the system has no heading stability, and if rotated rigidly would continue on its new heading as if undisturbed. Stability of the remaining roots requires that

$$\hat m<1 \quad (16)$$

and

$$\hat\ell<\hat\ell_c \quad (17)$$

where $\hat\ell_c$ is defined in Eq. (13). As before, condition (17) assures "static" stability while Eq. (16) guarantees "dynamic" stability. Stable values of $\hat m$ and $\hat\ell$ for fixed values of (D/W) are given in Fig. 5. Since condition (17) is the same as Eq. (13), the curved boundaries (hyperbolas) in Fig. 5 are identical to those in Fig. 2.

Discussion

The principal new feature of the present analysis is the inclusion of helicopter motions. Since other studies have not included these degrees of freedom, it is not possible to make direct comparisons. However, it can be observed that our results differ qualitatively from other predictions based on aerodynamic instabilities. For example, in Ref. 4 it is reported that aerodynamic instabilities may result if cable lengths are too short, whereas, our analysis predicts that a length shorter than a critical value is needed for stability. A numerical example is helpful in clarifying the matter. Consider a CH-54 with $M=620$ slugs, carrying an empty conex box (8'×8'×20') with $m=46.6$ slugs. Take $C_D=1$ and $v^e=100$ knots; it is found that $(D/W)=1.54$ and $\hat\ell_c=11.1$. In dimensional terms the critical length is about 9800 ft! It seems typical that the critical length is very long; certainly beyond the validity of the assumption of negligible tether weight. Of course, for a light, bulky load ((D/W) very large), practical instability may arise. From Ref. 4 it is estimated that aerodynamic stability would be assured with tether lengths greater than 100 ft. Thus, the two bounds are widely separated.

It is instructive to consider what we predict for the stability of a given configuration as the speed (v^e) is increased. Since v^e is used in the nondimensionalization, a conclusion is not immediate. In terms of dimensional quantities

$$\ell_c=[(M+m)/K_L]\ [(K_L v^{e2}/mg)^2+1]^{1/2}/(K_L v^{e2}(mg) \quad (18)$$

As v^e increases with all other parameters constant, ℓ_c decreases. However, ℓ_c does not decrease to zero, but approaches an asymptotic limit given by $\ell_c=(M+m)/K_L$. Even for the case of light and bulky loads, this minimum critical length (i.e., minimum length for which there is an upper bound on speed) is quite large.

Conclusions

It has been shown that for small motions about the equilibrium condition, the dynamical equations uncouple into longitudinal and lateral sets. For a special case ($b=1$) longitudinal stability requires: that the ratio of the mass of the load to that of the helicipter ($\hat m$) be less than a function of the drag-to-weight ratio (D/W); and that the length of the tether be less than a function of $\hat m$ and D/W. The general conditions for lateral stability are similar to, but more stringent than, the special longitudinal requirements. In particular, the tether-length restriction is identical to the result for the special ($b=1$) longitudinal case. However, for lateral stability the mass ratio must be less than one. Although values of $\hat m$ greater than unity are not currently attained, it may be that future designs will fall in this unstable region. Clearly, it would be worthwhile to investigate more complete dynamical models to determine the extent of the instability and possible cures. Additionally, one may wish to formulate a more reasonable aerodynamic model of the helitopter, including rotor dynamics. To do so would, in general, require the inclusion of rigid body motions of the helicopter so that the composite would be a high-order dynamic model (about twentieth order).

Appendix

The pupose of this section is to indicate more fully how the dynamical equations were derived. Briefly, we shall write the equations of motion separately for the helicopter and the load, and then eliminate the unknown tension in the tether (R). Flight-path coordinates are used for the helicopter with unit vectors: $\hat e_t$ tangent to the flight path; $\hat e_n$ normal to the flight path in the vertical plane ($\gamma\neq\pi/2$); and $\hat e_o$ normal to the flight path in the horizontal plane. The component equations are:

$$M\dot v=F_t-K_H v^2-Mg\sin\gamma$$
$$-R[\sin\theta\cos\gamma\cos(\lambda-\phi)-\sin\gamma\cos\theta] \quad (A1)$$

$$Mv\dot{\gamma}=F_n-Mg\cos\gamma + R[\sin\theta\sin\gamma\cos(\lambda-\phi)+\cos\gamma\cos\theta] \quad \text{(A2)}$$

and

$$Mv\cos\gamma\dot{\phi}=F_o-R\sin\theta\sin(\lambda-\phi) \quad \text{(A3)}$$

The equations of motion for the load are written in terms of components along axes (ξ, η, ζ).

$$m\ddot{\xi}=D_{L\xi}+R\sin\theta\cos\lambda \quad \text{(A4)}$$

$$m\ddot{\eta}=D_{L\eta}+R\sin\theta\sin\lambda \quad \text{(A5)}$$

$$m\ddot{\zeta}=D_{L\zeta}+R\cos\theta+mg \quad \text{(A6)}$$

Here $D_{L\xi}$, $D_{L\eta}$, and $D_{L\zeta}$ are the components of the drag on the load along the coordinate axes. The Cartesian coordinates (ξ, η, ζ) of the load are related kinematically to the Cartesian coordinates of the helicipter (x, y, z) and the pendulum angles θ and λ. For the *velocities* we have

$$\dot{\xi}=v\cos\gamma\cos\phi+\ell[\dot{\theta}\cos\theta\cos\lambda-\dot{\lambda}\sin\theta\sin\lambda] \quad \text{(A7)}$$

$$\dot{\eta}=v\cos\gamma\sin\phi+\ell[\dot{\theta}\cos\theta\sin\lambda+\dot{\lambda}\sin\theta\cos\lambda] \quad \text{(A8)}$$

$$\dot{\zeta}=-v\sin\gamma-\ell\dot{\theta}\sin\theta \quad \text{(A9)}$$

The procedure now is to substitute the expressions for R and the various trigonometric functions in Eqs. (A4-A6) into the corresponding places in Eqs. (A1-A3). For example, the term $R\sin\theta\sin\lambda$ which appears in Eq. (A1)** is given by $m\ddot{\xi}-D_{L\xi}$ from Eq. (A4). Subsequently, the derivatives of (ξ, η, ζ) can be written in terms of the required variables $(v, \gamma, \theta, \lambda, \phi)$ from Eqs. (A7-A9). The results are

$$\begin{aligned}&(M+m)\dot{v}-F_t+K_Hv^2+(M+m)g\sin\gamma\\&+m\ell\{\cos\gamma[\cos(\lambda-\phi)(\ddot{\theta}\cos\theta-(\dot{\theta}^2+\dot{\lambda}^2)\sin\theta)\\&-\sin(\lambda-\phi)(2\dot{\theta}\dot{\lambda}\cos\theta+\ddot{\lambda}\sin\theta)]\\&+\sin\gamma[\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta]\}\\&+K_Lv_L\{v+\ell\cos\gamma[\dot{\theta}\cos\theta\cos(\lambda-\phi)\\&-\dot{\lambda}\sin\theta\sin(\lambda-\phi)]+\ell\dot{\theta}\sin\gamma\sin\theta\}=0\end{aligned} \quad \text{(A10)}$$

$$\begin{aligned}&(M+m)v\dot{\gamma}-F_n+(M+m)g\cos\gamma-m\ell\{\sin\gamma[\cos(\lambda-\phi)\\&(\ddot{\theta}\cos\theta-(\dot{\lambda}^2+\dot{\theta}^2)\sin\theta)-\sin(\lambda-\phi)\\&\times(\ddot{\lambda}\sin\theta+2\dot{\theta}\dot{\lambda}\cos\theta)]-\cos\gamma[\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta]\}\\&-K_Lv_L\ell\{\sin\gamma[\dot{\theta}\cos\theta\cos(\lambda-\phi)\\&-\dot{\lambda}\sin\theta\sin(\lambda-\phi)]-\dot{\theta}\sin\theta\cos\gamma\}=0\end{aligned} \quad \text{(A11)}$$

$$\begin{aligned}&(M+m)v\cos\gamma\dot{\phi}-F_o+m\ell\{\sin(\lambda-\phi)\\&\times[\ddot{\theta}\cos\theta-(\dot{\theta}^2+\dot{\lambda}^2)\sin\theta]+\cos(\lambda-\phi)\\&[\ddot{\lambda}\sin\theta+2\dot{\theta}\dot{\lambda}\cos\theta]\}+K_Lv_L\ell\\&\{\dot{\theta}\cos\theta\sin(\lambda-\phi)+\dot{\lambda}\sin\theta\cos(\lambda-\phi)\}=0\end{aligned} \quad \text{(A12)}$$

In these equations v_L is the speed of the load [equal to $(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)^{1/2}$] and is given by

$$\begin{aligned}v_L=\{&v^2+\ell^2(\dot{\theta}^2+\dot{\lambda}^2\sin^2\theta)+2v\ell[\dot{\lambda}\sin\theta\cos\gamma\sin(\phi-\lambda)\\&+\dot{\theta}\cos\gamma\cos\theta\cos(\phi-\lambda)+\dot{\theta}\sin\gamma\sin\theta]\}^{1/2}\end{aligned} \quad \text{(A13)}$$

**The trigonometric functions ' cos $(\lambda-\phi)$' must first be expanded.

Two additional independent equations can be derived by eliminating R among equations (A4-A6). First we multiply Eqs. (A4) by sin λ, Eq. (A5) by cos λ and substract. There results

$$\begin{aligned}&m\{\sin[\lambda-\phi][\dot{v}\cos\gamma-v\dot{\gamma}\sin\gamma]\\&-\cos(\lambda-\phi)v\dot{\phi}\cos\gamma-\ell[2\dot{\theta}\dot{\lambda}\cos\theta+\ddot{\lambda}\sin\theta]\}\\&K_Lv_L\{v\cos\gamma\sin(\lambda-\phi)-\ell\dot{\lambda}\sin\theta\}=0\end{aligned} \quad \text{(A14)}$$

Finally, we may solve explicitly for R by using Eqs. (A4) and (A5) to obtain $R\sin\theta$ and then using Eq. (A6) along with the identity $\sin^2+\cos^2=1$.

The resulting expression for R can then be used in any of Eqs. (A4-A6). In particular, from Eq. (A6) we find

$$\begin{aligned}&m\{\cos\theta[\cos(\lambda-\phi)(\dot{v}\cos\gamma-v\dot{\gamma}\sin\gamma)\\&+\sin(\lambda-\phi)v\cos\gamma\dot{\phi}-\ell\dot{\lambda}^2\sin\theta]\\&+\sin\theta[\dot{v}\sin\gamma+v\dot{\gamma}\cos\gamma]+\ell\ddot{\theta}\}\\&+mg\sin\theta+K_Lv_L\{v[\cos\gamma\cos(\lambda-\phi)\cos\theta\\&+\sin\gamma\sin\theta]+\ell\dot{\theta}\}=0\end{aligned} \quad \text{(A15)}$$

The general motion is governed by the five equations (A10-A12, A14 and A15), where the quantity v_L is given by Eq. (A13).

For those with a bent toward analytical mechanics we observe that Eqs. (A10-A12) would result from an application of $F=MA$ to the center of mass of the helicopter/load system, while Eqs. (A14) and (A15) follow from angular momentum about the center of mass.††

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††Since the torque and moment of inertia about the tether axis are zero, only two nontrivial scalar equations result.